$$A^{\pm} = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin\varphi\cos^2\varphi\sin\Phi}{\omega^{3/2}(\eta,\varphi)} \exp R^{\pm}(\cos R^{\pm} + \sin R^{\pm})\,d\Phi\,d\varphi$$

$$R^{\pm} = Rl^{-1}\omega^{1/2}(\eta,\varphi)\,|m^{\pm}|$$

a) When $R \ll l$, we have from (4.6)

$$v_\varphi \sim M\nu_0^{-3/2}\omega_0^{-1/2}R^2l^{-2}\sin\theta\cos\theta F(\eta) \tag{4.7}$$

where $F(\eta)$ is a monotonically increasing function, $F(\eta) \sim \eta$ when $\eta \ll 1$ and $F(\eta) \sim \sqrt{\eta}$ when $\eta \gg 1$.

b) When $R \gg l$ within the bounds of the region of the main cell $r \gg z \gg l$ (4.4), for v_φ from (4.6) we have the asymptotic form

$$-v_\varphi \sim M2^{-1/2}\nu_0^{-3/2}\omega_0^{-1/2}R^{-1}l\operatorname{ctg}(\theta)P(\eta) \tag{4.8}$$

where $P(\eta)$ is well approximated by the function $P(\eta) \sim \eta(1+\eta^2)^{-1}$. The slower nature of the decrease in the azimuthal velocity (4.8) in this region than in the case of the radial and vertical velocities (4.4) should be noted. Formulae (4.7) and (4.8) enable one to establish the dependence of the intensity of the twisting $v_\varphi(r, z)$ on the radius when $z = \text{const}$ in the main cell. When $R \gg l$, the motion is close to the rotation of a solid body ($v_\varphi \sim r$) while, when $R \gg l$, it is close to a potential vortex ($v_\varphi \sim 1/r$).

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DEPENDENCE OF THE DISPERSION CURVES OF INTERNAL WAVES OF A STRATIFIED OCEAN ON THE VAISALA-BRENT FREQUENCY*

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Under the condition that the minimum of the Vaisala-Brent frequency (VBF) is greater than the Coriolis parameter, a parametric form of the dispersion curves of the internal gravitational waves in an ocean of constant depth with continuously variable VBF is obtained. This form is used when obtaining estimates of the dc displacements as a function of the VBF displacement and when isolating the VBF which admit of a unique restoration from a sequence of dispersion curves.

1. Formulation of the problem. We consider a horizontal continuously stratified ocean of constant depth $\mathbf{H}$. Its upper surface is the x/y plane, and the z axis is directed vertically upwards. The dispersion curves of the internal gravitational waves are found /1/ as the eigenvalues $\omega^2 = \omega_n^2(k^2)$ of the boundary value problem

$$W'' - \frac{\mu(z)}{g}W' + \frac{\mu(z)-\omega^2}{\omega^2-f^2}k^2W = 0, \quad W(-\mathbf{H}) = W(0) = 0 \tag{1.1}$$

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where $\mu(z)$ is the VBF squared, f is the Coriolis parameter, g is the acceleration due to gravity, k is the wave number, and ω is the frequency of the free harmonic waves. When (1.1) is obtained in /1/, $W(z)$ is understood to be the amplitude of the vertical velocity component of the fluid particles. In spectral analysis of problem (1.1), the meaning of $W(z)$ is a matter of indifference. For convenience, we shall assign to $W(z)$ the dimensions of length, i.e., the same as for the variable z.

It was shown in /1/ that, under the natural condition $\min\{\mu(z): z\in[-\mathbf{H},0]\}>f^2$, there is a denumerable system of eigencurves $\{\omega^2=\omega_n^2(k^2)\}$, the functions $\omega_n^2(k^2)$ are positive and increase as k^2 increases, while

$$\lim_{k\to 0}\omega_n^2(k^2)=f^2,\quad \lim_{k\to\infty}\omega_n^2(k^2)=\max_{[-\mathbf{H},0]}\mu(z)$$

and we have the asymptotic form

$$\omega_n^2(k^2)=f^2+k^2/\lambda_n^\circ+O(k^4),\quad k\to 0 \tag{1.2}$$

where λ_n° is the n-th eigenvalue of the boundary value problem

$$(\rho_*W')'+\lambda^0(\mu(z)-f^2)\rho_*W=0,\quad W(-\mathbf{H})=W(0)=0 \tag{1.3}$$

$$\rho_*(z)=\exp\left[g^{-1}\int_z^0\mu(\xi)\,d\xi\right]$$

Problem (1.3) has a denumerable sequence of positive and simple eigenvalues $\lambda_1^0<\lambda_2^0<\ldots$.

Our aim is to obtain the parametric form of the dispersion curve, and to use it when considering the converse spectral problem and for estimating the displacement of $\omega_n^2(k^2)$ as a function of the VBF displacement.

2. Construction of Green's function and reduction of problem (1.1) to an integral equation. In the same way as in /1/, we can transform from problem (1.1) to the problem

$$\begin{gathered}(\rho_*W')'+\lambda(\mu(z)-f^2)\rho_*W-k^2\rho_*W=0\\ W(-\mathbf{H})=W(0)=0,\quad \lambda=k^2(\omega^2-f^2)^{-1}\end{gathered} \tag{2.1}$$

Since the equation

$$(\rho_*W')'+(\lambda\mu-\lambda^2g^2)\rho_*W=0$$

has the solutions /2/

$$W_1^*(z)=\int_{-\mathbf{H}}^{z}\frac{1}{\rho_*(z)}e^{\lambda g(z-2x)}dx,\quad W_2^*(z)=\int_z^0\frac{1}{\rho_*(z)}e^{\lambda g(z-2x)}\,dx \tag{2.2}$$

it is convenient to write (2.1) as

$$(\rho_*W')'+(\lambda\mu(z)-\lambda^2g^2)\rho_*W=(-\lambda^2g^2+\lambda f^2+k^2)\rho_*W,$$

and to transform from the parameters λ and k to the parameters λ and s, where $s=\lambda^2g^2-\lambda f^2-k^2$.

Instead of problem (2.1) we will consider the corresponding problem

$$\begin{gathered}(\rho_*W')'+(\lambda\mu(z)-\lambda^2g^2)\rho_*W=-s\rho_*W\\ W(-\mathbf{H})=W(0)=0\end{gathered} \tag{2.3}$$

It is easily verified directly that $s=0$ is not an eigenvalue of problem (2.3) with any real λ. Hence, using the solutions (2.2), we can construct for problem (2.3) Green's function

$$G(z,t,\lambda)=\frac{-1}{\mathbf{a}(\lambda)}\begin{cases}W_1^*(z)W_2^*(t), & z\leqslant t\\ W_1^*(t)W_2^*(z), & z\geqslant t\end{cases} \tag{2.4}$$

$$\mathbf{a}(\lambda)=\int_{-\mathbf{H}}^0\frac{1}{\rho_*(z)}e^{-2\lambda gz}\,dz$$

Boundary value problem (2.3) is equivalent to the integral equation

$$y(z)=s\int_{-\mathbf{H}}^0 K(z,t,\lambda)y(t)\,dt \tag{2.5}$$

$$y(z)=\sqrt{\rho_*(z)}\,W(z),\quad K(z,t,\lambda)=-\sqrt{\rho_*(z)\rho_*(t)}\,G(z,t,\lambda)$$

3. Parametric equations of the dispersion curves. The oscillation conditions /3/ hold for the kernel $K(z, t, \lambda)$ with any real λ. Hence integral Eq.(2.5) has a denumerable set of positive and simple eigenvalues $s_1(\lambda) < s_2(\lambda) < \ldots$ for any real λ /3/.

The pair of equations

$$k^2(\omega^2 - f^2)^{-1} = \lambda, \; s_n(\lambda) = \lambda^2 g^2 - \lambda f^2 - k^2 \tag{3.1}$$

defines the dispersion curve $\omega^2 = \omega^2_{m_n}(k^2)$ in parametric form, provided the parameter λ runs over the set of values of the function

$$\lambda_n(k^2) = k^2(\omega^2_{m_n}(k^2) - f^2)^{-1}$$

considered on the ray $(0, +\infty)$. It was shown in /1/ that $d\lambda_n/dk^2 > 0$, $k^2 \in (0, +\infty)$. Hence the set of values of the function λ_n is the ray $(c_n, +\infty)$, where $c_n = \lim \lambda_n(k^2)$, $k \to 0$. It follows from the asymptotic form (1.2) that $c_n = \overset{\circ}{\lambda}_{m_n}$.

The sequence $\{c_n\}$ is increasing.

For, for fixed $\lambda > c_n$, one of the points of intersection of the lines

$$k^4g^2(\omega^2 - f^2)^{-2} - k^2f^2(\omega^2 - f^2)^{-1} - k^2 = s_n(\lambda) \tag{3.2}$$
$$k^2(\omega^2 - f^2)^{-1} = \lambda, \; k^2 > 0, \; \omega^2 > 0$$

in the $k^2\omega^2$ plane lies on the m_n-th dispersion curve. We introduce the notation $y = k^2$, $x = k^2(\omega^2 - f^2)^{-1}$. The lines (3.2) will intersect if and only if the lines

$$x = \lambda, \; y = g^2x^2 - f^2x - s_n(\lambda), \; x > 0, \; y > 0 \tag{3.3}$$

intersect in the xy plane.

The lines (3.3) will intersect at a unique point for those, and only those, λ for which we have

$$\lambda > F_n(\lambda), \quad F_n(\lambda) = (2g^2)^{-1}[f^2 + \sqrt{f^4 + 4g^2s_n(\lambda)}] \tag{3.4}$$

The number c_n is obviously defined as the value of λ for which the equals sign is obtained in (3.4). Since $c_n = F_n(c_n) < F_{n+1}(c_n)$, then $c_{n+1} > c_n$.

Since the sequence $\{c_n\}$, $\{\lambda_n^\circ\}$ are increasing, it follows that $m_n = n$.

On solving system (3.1) for k^2 and ω^2, we obtain the parametric equation of the dispersion curve $\omega^2 = \omega_n^2(k^2)$ in the form

$$k^2 = \lambda^2g^2 - \lambda f^2 - s_n(\lambda), \; \omega^2 = \lambda g^2 - \lambda^{-1}s_n(\lambda), \; \lambda \in (\lambda_n^\circ + \infty) \tag{3.5}$$

4. Specification of the dispersion curve in implicit form. It follows from (3.1) that the equation

$$s_n\left(\frac{k^2}{\omega^2 - f^2}\right) = \frac{g^2k^4}{(\omega^2 - f^2)^2} - \frac{f^2k^2}{\omega^2 - f^2} - k^2 \tag{4.1}$$

defines the n-dispersion curve in implicit form.

In the special case when $\mu(z) \equiv \mu_0$, we obtain

$$s_n(\lambda) = g^2\lambda^2 - \mu_0\lambda + n^2\pi^2\mathrm{H}^{-2} + {}^1/_4\,\mu_0^2g^{-2}$$

and Eq.(4.1) has the solution

$$\omega^2 = f^2 + k^2(\mu_0 - f^2)(k^2 + n^2\pi^2\mathrm{H}^{-2} + {}^1/_4\,\mu_0^2g^{-2})^{-1}$$

which is the same as the well-known representation of the n-th dispersion curve /1/.

The dependence of s_n on λ cannot be calculated explicitly in the general case; but it can be asserted that $s_n(\lambda)$ is a holomorphic function on the real axis /4/.

For, the family of integral operators $K(\lambda)$ with symmetric real kernel $K(z, t, \lambda)$ is a self-adjoint and holomorphic family of type (A) of compact operators, defined on the real axis, and, for any $\lambda \in R$, the operator $K(\lambda)$ has simple non-vanishing eigenvalues.

Some applications of the parametric form (3.5) of the dispersion curve are considered below.

5. Estimation of the shift of the frequency of free oscillations as a function of the VBF shift. We shall confine ourselves to functions $\mu(z)$ which belong to the parametric family Γ, consisting of polynomials of not higher than a fixed degree r

$$\sum_{j=0}^{r} c_jz^j, \quad c_j \in [c_1^{(j)}, \; c_2^{(j)}], \quad j = 0, 1, 2, \ldots, r$$

and satisfying the condition $m = \min\{\mu(z) : z \in [-\mathrm{H}, 0], \; \mu \in \Gamma\} > f^2$.

For an admissible value $c = (c_0, c_1, \ldots, c_r)$, we denote a function of the family Γ by $\mu(z, c)$. The kernel $K(z, t, \lambda) = K(z, t, \lambda, c)$ of the integral operator in (2.5) depends analytically on λ and c for real λ: $c \in \Pi = [c_1^{(0)}, c_2^{(0)}] + [c_1^{(1)}, c_2^{(1)}] \times \ldots \times [c_1^{(r)}, c_2^{(r)}]$. We will denote the eigenvalues of the integral operator with kernel $K(z, t, \lambda, c)$ by $s_n(\lambda, c)$. Arguing in the same way as above, we find that $s_n(\lambda, \mathbf{c})$ are holomorphic functions of λ and $\mathbf{c}$ in $R \times \Pi$. The corresponding eigenfunctions of boundary value problem (2.3), normalized by the condition

$$\int_{-\mathbf{H}}^{0} \rho_*(z) W_n^2(z, \lambda, \mathbf{c})\, dz = d^3$$

where d is numerically equal to unity and has the dimensions of length, also depends holomorphically on the parameters λ and $\mathbf{c}$ in $R \times \Pi$.

We consider the functions $\mu(z, \gamma) = \gamma_0 + \gamma_1 z + \ldots + \gamma_r z^r$ ($\gamma = \alpha, \beta$), which belong to the family Γ. Corresponding to these functions we have sequences of eigenvalues $\{s_n(\lambda, \gamma)\}$ and sequences of dispersion curves $\{\omega^2 = \omega_n^2(k^2, \gamma)\}$.

We obtain global and local estimates

$$|\Delta\omega_n^2| \leqslant \mathbf{A} \, \|\Delta\mu\|, \quad \Delta\mu = \mu(z, \alpha) - \mu(z, \beta)$$

$$\Delta\omega_n^2 = \omega_n^2(k^2, \alpha) - \omega_n^2(k^2, \beta)$$

with different values of $\mathbf{A}$ and the norm $\|\Delta\mu\|$ defined below.

We consider the equation

$$s_n(\lambda_1, \alpha) - s_n(\lambda_2, \beta) = g^2(\lambda_1^2 - \lambda_2^2) - f^2(\lambda_1 - \lambda_2) \tag{5.1}$$

with respect to real variables λ_1 and λ_2, with the same dimensionality as $k^2\omega^{-2}$. By (4.1), Eq. (5.1) has the solution

$$\lambda_1 = y_1 = k^2[\omega_n^2(k^2, \alpha) - f^2]^{-1}, \ \lambda_2 = y_2 = k^2[\omega_n^2(k^2, \beta) - f^2]^{-1}.$$

Subtracting and adding $s_n(\lambda_2, \alpha)$ to the left-hand side of (5.1), using Taylor's formula and identity transformations, we obtain the equation

$$\left[\frac{\partial s_n}{\partial \lambda}(\lambda_1, \alpha) + f^2\right](\lambda_1 - \lambda_2) - (g^2 - \delta)(\lambda_1 - \lambda_2)^2 - \tag{5.2}$$

$$g^2(\lambda_1^2 - \lambda_2^2) + \tau = 0, \quad \delta = g^2 - {}^1/_2 \frac{\partial^2 s_n}{\partial \lambda^2}(\lambda_1 + \theta_1(\lambda_2 - \lambda_1), \alpha)$$

$$\tau = s_n(\lambda_2, \alpha) - s_n(\lambda_2, \beta) =$$

$$\sum_{j=0}^{r} (\alpha_j - \beta_j) \frac{\partial s_n}{\partial c_j}(\lambda_2, \beta + \theta_2(\alpha - \beta)), \quad \theta_1, \theta_2 \in (0, 1)$$

To find $\partial s_n/\partial\lambda(\lambda, \mathbf{c})$, we put $s = s_n(\lambda, \mathbf{c})$, $\mu(z) = \mu(z, \mathbf{c})$, $W = W_n(z, \lambda, \mathbf{c})$ in Eq.(2.3), and differentiate the resulting identity with respect to λ. We then multiply both sides of the result by $W_n(z, \lambda, \mathbf{c})$ and integrate them over the interval $[-\mathbf{H}, 0]$. The result is

$$\frac{\partial s_n}{\partial \lambda}(\lambda, \mathbf{c}) = 2\lambda g^2 - \nu(\lambda, \mathbf{c}) - f^2 \tag{5.3}$$

$$\nu(\lambda, \mathbf{c}) = d^{-3} \int_{-\mathbf{H}}^{0} (\mu(z, \mathbf{c}) - f^2) \rho_*(z, \mathbf{c}) W_n^2(z, \lambda, \mathbf{c})\, dz$$

In the same way, we obtain

$$\frac{\partial s_n}{\partial c_j}(\lambda, \mathbf{c}) = -d^{-3} \int_{-\mathbf{H}}^{0} Q(z, \lambda, \mathbf{c}) W_n^2(z, \lambda, \mathbf{c})\, dz \tag{5.4}$$

$$Q(z, \lambda, \mathbf{c}) = \frac{1}{2g} \frac{\partial}{\partial z}(z^j \rho_*(z, \mathbf{c})) + \lambda z^j \rho_*(z, \mathbf{c})$$

Substituting (5.3) for $\partial s_n/\partial\lambda(\lambda_1, \alpha)$ into (5.2), we arrive at the equation

$$\delta(\lambda_1 - \lambda_2)^2 - \nu(\lambda_1, \alpha)(\lambda_1 - \lambda_2) + \tau = 0 \tag{5.5}$$

We distinguish two cases: 1) $\delta = 0$ ($\delta = 0$ in the case $\mu \equiv$ const), 2) $\delta \neq 0$ (it can be shown that $\delta > 0$ if $\mu \not\equiv$ const, $n = 1$). By (5.5), in the first case we have $\lambda_1 - \lambda_2 = \tau\nu^{-1}(\lambda_1, \alpha)$, and in the second $\lambda_1 - \lambda_2 = (2\delta)^{-1}[\nu(\lambda_1, \alpha) \pm \sqrt{\nu^2(\lambda_1, \alpha) - 4\delta\tau}]$.

We shall discuss case 2) in more detail. Henceforth, for λ_1, λ_2 we substitute y_1, y_2 respectively.

From (5.1) we have the expression for τ:

$$\tau = s_n(y_2, \alpha) - s_n(y_1, \alpha) - g^2(y_2^2 - y_1^2) + f^2(y_2 - y_1) \tag{5.6}$$

Using (5.3), we obtain

$$y_1 - y_2 = \tau \left[\frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \nu(\lambda, \alpha)\, d\lambda\right]^{-1} \tag{5.7}$$

From (5.7) we have

$$|y_1 - y_2| < \frac{|\tau|}{(m(\alpha) - f^2)}, \quad \alpha, \beta \in \Pi \qquad (5.8)$$
$$m(\alpha) = \min\{\mu(z, \alpha): z \in [-\mathbf{H}, 0]\}$$

We can dispense with the difference $m(\alpha) - f^2$ in the denominator if we confine ourselves to estimating $|y_1 - y_2|$ for fixed $\alpha \in \Pi$, and β fairly close to α.

Since $s_n(\lambda, \mathbf{c})$ is holomorphic with respect to λ and $\mathbf{c}$ in R and Π, and recalling (5.3), (5.4), it follows that the dispersion curves $\omega^2 = \omega_n^2(k^2, \mathbf{c})$ are holomorphic with respect to $\mathbf{c}$ in Π. Hence, for fixed k^2, the difference $y_1 - y_2$ tends to zero as $\beta \to \alpha$. Since, as $\beta \to \alpha$, we have $\tau \to 0$, $\nu(y_1, \alpha) \geqslant m^2 - f^2 > 0$, δ is bounded, then, as $\beta \to \alpha$,

$$y_1 - y_2 = (2\delta)^{-1}[\nu(y_1, \alpha) - \sqrt{\nu^2(y_1, \alpha) - 4\delta\tau}].$$

Since, for $\beta = \alpha$, we have $\nu^2(y_1, \alpha) - 4\delta\tau = \nu^2(y_1, \alpha) > 0$, there is a neighbourhood $\Pi(\alpha) = [\alpha_1 - \varepsilon_1, \alpha_1 + \varepsilon_1'] \times \ldots + [\alpha_r - \varepsilon_r, \alpha_r + \varepsilon_r'], \varepsilon_j > 0, \varepsilon_j' > 0, j = 0, 1, \ldots, r$ of the point α, lying in Π, such that, for $\beta \in \Pi(\alpha)$, we have $\nu^2(y_1, \alpha) - 4\delta\tau > 0$, and hence

$$|y_1 - y_2| < 2|\tau|\nu^{-1}(y_1, \alpha), \quad \beta \in \Pi(\alpha) \qquad (5.9)$$

We find an upper limit for $|\tau|$. Using Eq. (5.4) in the expression for τ, and noting that $\partial\rho_*/\partial z = -\mu\rho_*/g$, we obtain

$$\tau = d^{-3}\int_{-\mathbf{H}}^{0} \rho_*(z, \beta + \theta_2(\alpha - \beta))\left[\frac{1}{2g}\frac{\partial(\Delta\mu)}{\partial z} + y_2\Delta\mu - \frac{1}{2g^2}\mu(z, \beta + \theta_2(\alpha - \beta))\Delta\mu\right]W_n^2(z, y_2, \beta + \theta_2(\alpha - \beta))\,dz.$$

Putting

$$\|\Delta\mu\| = \max\left\{\max_{[-\mathbf{H}, 0]}|\Delta\mu|, d\max_{[-\mathbf{H}, 0]}\left|\frac{\partial(\Delta\mu)}{\partial z}\right|\right\},$$

we obtain from the previous equation the limit

$$|\tau| \leqslant \|\Delta\mu\|\left[\frac{1}{2gd} + \frac{\mathbf{M}(\beta + \theta_2(\alpha - \beta))}{2g^2} + y_2\right].$$
$$\mathbf{M}(\mathbf{c}) = \max\{\mu(z, \mathbf{c}): z \in [-\mathbf{H}, 0]\}$$

Since $\mu(z, \beta + \theta_2(\alpha - \beta)) = \theta_2\mu(z, \alpha) + (1 - \theta_2)\mu(z, \beta)$, then $\mathbf{M}(\beta + \theta_2(\alpha - \beta)) \leqslant \max\{\mathbf{M}(\alpha), \mathbf{M}(\beta)\} = \mathbf{M}(\alpha, \beta)$ and for $|\tau|$ we obtain

$$|\tau| \leqslant y_2\left[1 + \frac{1}{y_2}\left(\frac{1}{2gd} + \frac{1}{2g^2}\mathbf{M}(\alpha, \beta)\right)\right]\|\Delta\mu\|. \qquad (5.10)$$

To find the required lower limit for $\nu(y_1, \alpha)$, we put $\lambda = y_1$, $\mu(z) = \mu(z, \alpha)$, $W = W_n(z, y_1, \alpha)$ in Eq. (2.1). Multiplying both sides of the result by $W_n(z, y_1, \alpha)$ and then integrating the result over the interval $[-\mathbf{H}, 0]$, we find

$$\nu(y_1, \alpha) = \frac{1}{y_1}\left[k^2 + d^{-3}\int_{-\mathbf{H}}^{0}\rho_*(z, \alpha)W_n'^2(z, y_1, \alpha)\,dz\right] \geqslant \frac{k^2}{y_1} = \omega_n^2(k^2, \alpha) - f^2 \qquad (5.11)$$

By (5.8) and (5.10) we have

$$|\Delta\omega_n^2| \leqslant \frac{\omega_n^2(k^2, \alpha) - f^2}{m(\alpha) - f^2}\left[1 + \frac{\Gamma(\alpha, \beta)}{y_2}\right]\|\Delta\mu\| \qquad (5.12)$$
$$\Gamma(\alpha, \beta) = \frac{1}{2gd} + \frac{\mathbf{M}(\alpha, \beta)}{2g^2}$$

From (5.12) we obtain

$$|\Delta\omega_n^2| < \mathbf{A}(k, \alpha, \beta)\frac{\omega_n^2(k^2, \alpha) - f^2}{m(\alpha) - f^2}\|\Delta\mu\|$$
$$\mathbf{A}(k, \alpha, \beta) = 1 + k^{-2}(\mathbf{M}(\beta) - f^2)\Gamma(\alpha, \beta)$$

We can also obtain from (5.12) a limit which is independent of k:

$$|\Delta\omega_n^2| > \mathbf{B}(\alpha, \beta)\frac{\mathbf{M}(\alpha) - f^2}{m(\alpha) - f^2}\|\Delta\mu\|, \quad \alpha, \beta \in \Pi \qquad (5.13)$$
$$\mathbf{B}(\alpha, \beta) = 1 + \frac{\mathbf{H}}{8d}\left(1 + \frac{d\mathbf{M}(\alpha, \beta)}{g}\right)\exp[\mathbf{H}g^{-1}\mathbf{M}(\beta)] < \mathbf{B}$$

The limit (5.13) follows from the inequality $y_2 > \lambda_n^\circ(\beta)$, where $\lambda_n^\circ(\beta)$ is the n-th eigenvalue of problem (1.3) with $\mu(z) = \mu(z, \beta)$ and from the lower limit of $\lambda_n^\circ(\beta)$, obtained on reducing problem (1.3) to an integral equation.

From (5.9), (5.10), (5.11) we obtain

$$|\Delta\omega_n^2| < 2\mathbf{A}(k, \alpha, \beta)\,\|\Delta\mu\|, \quad |\Delta\omega_n^2| < 2\,\mathbf{B}(\alpha, \beta)\,\|\Delta\mu\|,$$
$$\beta \in \Pi(\alpha)$$

It follows from (5.13) that $|\Delta\omega_n^2|$ tends uniformly to zero with respect to k^2 in $(0, +\infty)$ and n in the set of integers as $\|\Delta\mu\| \to 0$.

Parametric families Γ of a different type can be similarly discussed, when the upper face of the ocean is a free surface.

6. Application of Green's function to the problem of restoring the VBF, given the dispersion law.

It is useful to specify Green's function explicitly when considering the possibility of uniquely restoring the function $\mu(z)$ from the sequence of dispersion curves. Notice that, to find $\mu(z)$, it suffices to be able to restore the function $\mathbf{a}(\lambda)$, apart from a constant factor, from the sequence of dispersion curves. For, in the case when $\mathbf{a}_1(\lambda) = \mathbf{A}\mathbf{a}(\lambda)$, where $\mathbf{A}$ is a non-zero constant, we have the representation

$$\mathbf{a}_1(\lambda) = \mathbf{A}\int_0^{\mathbf{H}} \frac{1}{\rho_*(t-\mathbf{H})} e^{-2\lambda g(t-\mathbf{H})}\,dt.$$

Using the expression for the inversion of the Laplace transform and the connection of the function ρ_* with μ, we obtain, for $t \in (0, \mathbf{H})$, $\sigma > 0$,

$$\mu(t-\mathbf{H}) = g\frac{d}{dt}\Big(\ln \int_{\sigma-i\infty}^{\sigma+i\infty} \mathbf{a}_1(\lambda)\, e^{2\lambda g(t-\mathbf{H})}\,d\lambda\Big) \tag{6.1}$$

Hence $\mu(z)$ is uniquely restored from the function $\mathbf{Aa}(\lambda)$ and the result of restoration is independent of the constant $\mathbf{A}$.

Let us show that $\mathbf{a}(\lambda)$ is sometimes uniquely defined apart from a constant factor by means of the first trace $\mathbf{A}_1(\lambda)$ of the integral operator with kernel $K(z, t, \lambda)$.

We can write for $\mathbf{A}_1(\lambda)$, using (2.4):

$$\mathbf{A}_1(\lambda) = \sum_{k=1}^{\infty} \frac{1}{s_k(\lambda)}, \quad \mathbf{A}_1(\lambda) = \int_{-\mathbf{H}}^{0} K(z, z, \lambda)\,dz = \frac{\mathbf{b}(\lambda)}{\mathbf{a}(\lambda)} \tag{6.2}$$
$$\mathbf{b}(\lambda) = \int_{-\mathbf{H}}^{0} \rho_*(z)\, W_1^*(z)\, W_2^*(z)\,dz$$

The function $\mathbf{A}_1(\lambda)$ is fully defined by the sequence of dispersion curves of the first of (6.2). This follows from (4.1) and the fact that $s_n(\lambda)$, $\mathbf{A}_1(\lambda)$ are analytic on the real axis.

From the second of (6.2), the function $\mathbf{a}(\lambda)$ is uniquely defined, apart from a constant factor, by $\mathbf{A}_1(\lambda)$, if $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ have no common zeros.

To sum up, if we know that $\mu(z)$ belongs to the set of functions for which $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ have no common zeros, then, given the sequence of dispersion curves, $\mu(z)$ can be uniquely restored by (6.1) from the class of curves corresponding to this set.

If $\mu(z) \equiv \mu_0$, then the functions

$$\mathbf{a}(\lambda) = \frac{\mathbf{H}\,\mathrm{sh}\,\Lambda}{\Lambda} e^{\Lambda}, \quad \mathbf{b}(\lambda) = \frac{\mathbf{H}^3 e^{\Lambda}}{2\Lambda^2}\Big[\mathrm{ch}\,\Lambda - \frac{\mathrm{sh}\,\Lambda}{\Lambda}\Big]$$
$$\Lambda = \mathbf{H}(\lambda g - {}^1\!/_2\,\mu_0/g)$$

have no common zeros.

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